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Periodic blow-up solutions and their limit forms for the generalized Camassa–Holm equation

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Abstract

In this paper, we consider the generalized Camassa–Holm equation

$$u_t + 2ku_x - u_{xxt} + au^2u_x = 2u_xu_{xx} + uu_{xxx}.$$

Under substitution $\xi = x - ct$, some new explicit periodic wave solutions and their limit forms are presented through some special phase orbits. These periodic wave solutions tend to infinity on $\xi - u$ plane periodically. Thus we call them periodic blow-up solutions. To our knowledge, such periodic blow-up solutions have not been found in any other equations.

Keywords: Generalized Camassa–Holm equation; Explicit solutions; Periodic blow-up

1. Introduction and main results

In 1993, Camassa and Holm [1] derived a shallow water wave equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad (1)$$

which is called Camassa–Holm equation or CH equation. Eq. (1) also was derived by Dai [2] as a model equation in hyperelastic rods.

For $k = 0$, Camassa and Holm [1] showed that Eq. (1) has peakons of the form $u(x,t) = c\,e^{-|x-ct|}$. For the case of $k \neq 0$ and the wave speed $c = \frac{k}{2}$, Liu and Qian [3] gave three ways to seek the peakon of Eq. (1). For any parameter k and constant wave speed c, Liu et al. [4] showed that Eq. (1) has peakons of the form

$$u(x,t) = (k + c)\,e^{-|x-ct|} - k, \quad (2)$$

which can be seen as a weak solution being similar to that in Ref. [5–7]. In Ref. [8–13] the blow-up phenomena of Eq. (1) were investigated. In Ref. [14] Liu et al. found two new bounded waves, the compacton-like wave and the kink-like wave, for Eq. (1).

In 2001, Dullin, Gottwald and Holm [15] presented a non-linear equation

$$u_t + c_0u_x + 3uu_x - \alpha^2(u_{xxt} + uu_{xxx} + 2u_xu_{xx}) + \gamma u_{xxx} = 0. \quad (3)$$

Clearly, when $\alpha^2 = 1$ and $\gamma = 0$, Eq. (3) becomes Eq. (1). In Refs. [16–18], it was shown that Eqs. (1) and (3) have many similar properties.

In 2001, Liu and Qian [19] suggested a generalized Camassa–Holm equation

$$u_t + 2ku_x - u_{xxt} + au^mu_x = 2u_xu_{xx} + uu_{xxx}. \quad (4)$$

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Tian and Song [20] gave some new peaked solitary wave solutions for Eq. (4) when $m = 1, 2, 3$. Khuri [21] gave some explicit expressions of the peakons and discontinuous solitary waves for Eq. (4) when $m = 1, 2, 3$. Shen and Xu [22] showed that Eq. (4) has compactons and cusp waves for arbitrary positive integer m. When $m = 2$, Eq. (4) becomes

$$u_t + 2ku_x - u_{xxt} + au^2u_x = 2u_xu_{xx} + uu_{xxx}. \tag{5}$$

For the case of $a = 3$ and $k = 0$, using several special functions, Wazwaz [23,24] obtained many explicit solitary wave solutions.

Liu and Ouyang [25] showed that the bell-shaped solitary wave and peakon coexist in Eq. (5) when $a = 3$ and $k = 0$.

In this paper, we consider Eq. (5). Through some special phase orbits, a new class of explicit periodic wave solutions is obtained. Since such solutions blow up periodically, they are called periodic blow-up solutions. Also the limit forms of these solutions are got.

In order to state our main results conveniently, for given constant c, let

$$l_1: \quad k = l_1(a,c) = \frac{c(3-ac)}{6}, \tag{6}$$

$$l_2: \quad k = l_2(a,c) = \frac{c(6-ac)}{12}, \tag{7}$$

$$\alpha = \sqrt{\frac{6c - 12k - ac^2}{a}} \quad \text{for} \quad \frac{6c-12k-ac^2}{a} > 0, \tag{8}$$

$$\beta_1 = \sqrt{\frac{|a|\alpha}{12}}, \tag{9}$$

$$\beta_2 = \sqrt{\frac{|ac|}{12}}, \tag{10}$$

$$\beta_3 = \sqrt{\frac{|a|(\alpha+|c|)}{24}}, \tag{11}$$

$$\beta_4 = \sqrt{\frac{|ac|}{24}}, \tag{12}$$

$\mathrm{sn}\, z = \mathrm{sn}(z, l)$ be the Jacobian elliptic function with modulus l, $\sec z$ and $\csc z$ be trigonometric functions, $\coth z$ and $\mathrm{csch}\, z$ be hyperbolic functions. On the parametric plane a–k, we mark the locations of the l_i $(i = 1, 2)$ and regions $(A_j), (B_j)$ $(j = 1-4)$ surrounded by l_i and k-axis as Fig. 1.

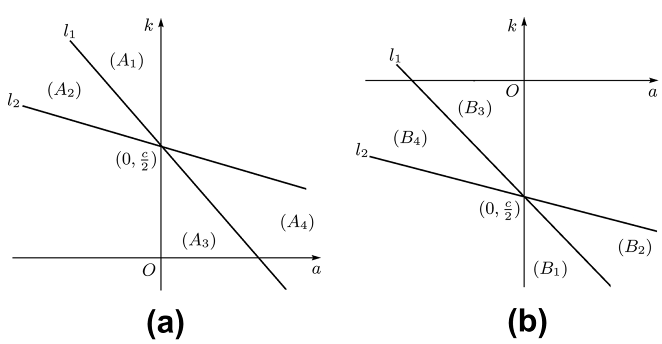

Fig. 1. The locations of l_i $(i = 1, 2)$ and regions $(A_j), (B_j)$ $(j = 1-4)$ on $a - k$ plane. (a) for given $c > 0$; (b) for given $c < 0$.

Using the notations above, our main results are stated in the following Propositions 1, 2 and Properties 1, 2.

Proposition 1. *For given constant $c > 0$ and parametric regions marked in Fig. 1a, on the solutions of Eq. (5) we have:*

(1) If $(a,k) \in (A_1)$, then there is a periodic blow-up solution

$$u_1(x,t) = \alpha(1 - 2\,\mathrm{sn}^{-2}\beta_1(x - ct)), \tag{13}$$

where the modulus of sn is

$$k_1 = \sqrt{\frac{\alpha + |c|}{2\alpha}}. \tag{14}$$

(2) If $a < 0$ and $(a,k) \in l_1$, then there is a blow-up solution

$$u_2(x,t) = c(1 - 2\coth^2\beta_2(x - ct)). \tag{15}$$

(3) If $(a,k) \in (A_2)$, then there is a periodic blow-up solution

$$u_3(x,t) = \alpha - (\alpha + |c|)\,\mathrm{sn}^{-2}\beta_3(x - ct), \tag{16}$$

where the modulus of sn is

$$k_2 = \sqrt{\frac{2\alpha}{\alpha + |c|}}. \tag{17}$$

(4) If $a < 0$ and $(a,k) \in l_2$, then there is a periodic blow-up solution

$$u_4(x,t) = -c\,\csc^2\beta_4(x - ct). \tag{18}$$

(5) If $(a,k) \in (A_3)$, then there is a periodic blow-up solution

$$u_5(x,t) = \alpha(2\,\mathrm{sn}^{-2}\beta_1(x - ct) - 1), \tag{19}$$

where the modulus of sn is

$$k_3 = \sqrt{\frac{\alpha - |c|}{2\alpha}}. \tag{20}$$

(6) If $a > 0$ and $(a,k) \in l_1$, then there is a periodic blow-up solution

$$u_6(x,t) = c\big(2\,\csc^2\beta_2(x - ct) - 1\big). \tag{21}$$

(7) If $(a,k) \in (A_4)$, then there is a periodic blow-up solution

$$u_7(x,t) = (\alpha + |c|)\,\mathrm{sn}^{-2}\beta_3(x - ct) - |c|, \tag{22}$$

where the modulus of sn is

$$k_4 = \sqrt{\frac{|c| - \alpha}{|c| + \alpha}}. \tag{23}$$

(8) If $a > 0$ and $(a,k) \in l_2$, then there is a blow-up solution

$$u_8(x,t) = c\,\mathrm{csch}^2\beta_4(x - ct). \tag{24}$$

Proposition 2. *For given constant $c<0$ and parametric regions marked in Fig. 1b, on the solutions of Eq. (5) we have:*

(1) If $(a,k)\in(B_1)$, then there is a periodic blow-up solution $-u_1(x,t)$.
(2) If $a>0$ and $(a,k)\in l_1$, then there is a blow-up solution $u_2(x,t)$.
(3) If $(a,k)\in(B_2)$, then there is a periodic blow-up solution $-u_3(x,t)$.
(4) If $a>0$ and $(a,k)\in l_2$, then there is a periodic blow-up solution $u_4(x,t)$.
(5) If $(a,k)\in(B_3)$, then there is a periodic blow-up solution $-u_5(x,t)$.
(6) If $a<0$ and $(a,k)\in l_1$, then there is a periodic blow-up solution $u_6(x,t)$.
(7) If $(a,k)\in(B_4)$, then there is a periodic blow-up solution $-u_7(x,t)$.
(8) If $a<0$ and $(a,k)\in l_2$, then there is a blow-up solution $u_8(x,t)$.

Property 1. *For given constant $c>0$, the solution $u_i(x,t)$ $(i=1-8)$ has the following relations:*

(1°) When $(a,k)\in(A_1)$ and tends to l_1, $u_1(x,t)$ becomes $u_2(x,t)$.
(2°) When $(a,k)\in(A_2)$ and tends to l_1, $u_3(x,t)$ becomes $u_2(x,t)$. When $(a,k)\in(A_2)$ and tends to l_2, $u_3(x,t)$ becomes $u_4(x,t)$.
(3°) When $(a,k)\in(A_3)$ and tends to l_1, $u_5(x,t)$ becomes $u_6(x,t)$.
(4°) When $(a,k)\in(A_4)$ and tends to l_1, $u_7(x,t)$ becomes $u_6(x,t)$. When $(a,k)\in(A_4)$ and tends to l_2, $u_7(x,t)$ becomes $u_8(x,t)$.

Property 2. *For given constant $c<0$, the solutions $u_i(x,t)$ $(i=1-8)$ have the following relations:*

(1*) If $(a,k)\in(B_1)$ and tends to l_1, then $-u_1(x,t)$ becomes $u_2(x,t)$.
(2*) If $(a,k)\in(B_2)$ and tends to l_1, then $-u_3(x,t)$ becomes $u_2(x,t)$. If $(a,k)\in(B_2)$ and tends to l_2, then $-u_3(x,t)$ becomes $u_4(x,t)$.
(3*) If $(a,k)\in(B_3)$ and tends to l_1, then $-u_5(x,t)$ becomes $u_6(x,t)$.
(4*) If $(a,k)\in(B_4)$ and tends to l_1, then $-u_7(x,t)$ becomes $u_6(x,t)$. If $(a,k)\in(B_4)$ and tends to l_2, then $-u_7(x,t)$ becomes $u_8(x,t)$.

Remark 1. When $c=0$, $u_i(x,t)$ $(i=1,3,5,7)$ become stationary solutions, and $u_j(x,t)$ $(j=2,4,6,8)$ become trivial solutions of Eq. (5).

Remark 2. In Propositions 1, 2 and Properties 1, 2, given $c>0$ or $c<0$, then we determine the lines l_1 and l_2. If given a and k, then under parametric condition $ak<0$, we have:

(1) If the wave speed c satisfies

$$c=\frac{1}{2a}\left(3-\sqrt{9-24ak}\right), \tag{25}$$

then Eq. (5) has a blow-up solution $u_2(x,t)$.
(2) If the wave speed c satisfies

$$c=\frac{1}{a}\left(3-\sqrt{9-12ak}\right), \tag{26}$$

then Eq. (5) has two periodic blow-up solutions $u_4(x,t)$ and

$$u_4^*(x,t)=-c\sec^2\beta_4(x-ct). \tag{27}$$

(3) If the wave speed c satisfies

$$c=\frac{1}{2a}\left(3+\sqrt{9-24ak}\right), \tag{28}$$

then Eq. (5) has two periodic blow-up solutions $u_6(x,t)$ and

$$u_6^*(x,t)=c\left(2\sec^2\beta_2(x-ct)-1\right). \tag{29}$$

(3) If the wave speed c satisfies

$$c=\frac{1}{a}\left(3+\sqrt{9-12ak}\right), \tag{30}$$

then Eq. (5) has a blow-up solution $u_8(x,t)$. If $\xi=x-ct$, then $u_i(x,t)$ becomes $u_i(\xi)$ $(i=1-8)$. For given c and (a,k) satisfying the conditions in Propositions 1 or 2, we can use computer to draw the graphs of $u_i(\xi)$ $(i=1-8)$.

Example 1. Letting $c=1$ and $a=-1$, then from (6) and (7) it follows that $l_1(-1,1)=2/3$ and $l_2(-1,1)=7/12$. Taking $k=0.8,2/3,0.6$ and $7/12$, respectively, then it is seen that $(a,k)=(-1,0.8)\in(A_1)$, $(a,k)=(-1,2/3)\in l_1$, $(a,k)=(-1,0.6)\in(A_2)$ and $(a,k)=(-1,7/12)\in l_2$. Substituting these data into the expressions of $u_i(\xi)$ $(i=1-4)$, on $\xi-u$ plane we draw their graphs as Fig. 2a, b, c and d.

If let $c=1$ and $a=1$, then similarly we get $l_1(1,1)=1/3$ and $l_2(1,1)=5/12$. Taking $k=0.1,1/3,0.41$ and $5/12$, then it is easy to see that $(a,k)=(1,0.1)\in(A_3)$, $(a,k)=(1,1/3)\in l_1$, $(a,k)=(1,0.41)\in(A_4)$ and $(a,k)=(1,5/12)\in l_2$. Substituting these data into the expressions of $u_i(\xi)$ $(i=5-8)$, on $\xi-u$ plane we draw their graphs as Fig. 2e, f, g and h. From Fig. 2 one can see visually that $u_2(\xi)$ and $u_8(\xi)$ blow up at $\xi=0$, and others blow up periodically.

2. Preliminary

In order to derive the expressions of solutions above, we establish a planar system corresponding to Eq. (5) and draw its bifurcation phase portraits.

For given constant c, substituting

$$\xi=x-ct \tag{31}$$

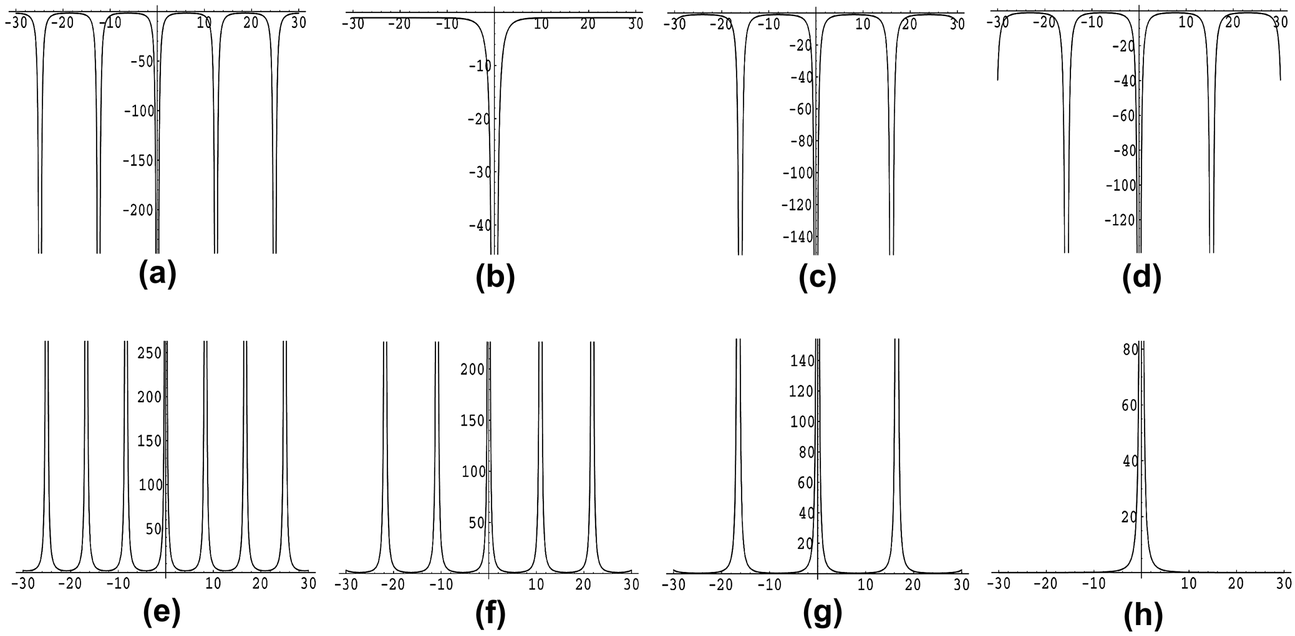

Fig. 2. The graphs of $u_i(\xi)$ $(i = 1 - 8)$ when $c = 1$ on $\xi - u$ plane. (a) graph of $u_1(\xi)$ for $(a,k) = (-1, 0.8) \in (A_1)$; (b) graph of $u_2(\xi)$ for $(a,k) = (-1, 2/3) \in l_1$; (c) graph of $u_3(\xi)$ for $(a,k) = (-1, 0.6) \in (A_2)$; (d) graph of $u_4(\xi)$ for $(a,k) = (-1, 7/12) \in l_2$; (e) graph of $u_5(\xi)$ for $(a,k) = (1, 0.1) \in (A_3)$; (f) graph of $u_6(\xi)$ for $(a,k) = (1, 1/3) \in l_1$; (g) graph of $u_7(\xi)$ for $(a,k) = (1, 0.41) \in (A_4)$; (h) graph of $u_8(\xi)$ for $(a,k) = (1, 5/12) \in l_2$.

and $u = \varphi(\xi)$ into Eq. (5), it follows that

$$-c\varphi' + 2k\varphi' + c\varphi''' + a\varphi^2\varphi' = 2\varphi'\varphi'' + \varphi\varphi'''. \tag{32}$$

Integrating (32) once and letting the integral constant be zero, we have

$$\varphi''(\varphi - c) = (2k - c)\varphi + \frac{a}{3}\varphi^3 - \frac{(\varphi')^2}{2}. \tag{33}$$

Via (33), we establish the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{\frac{a}{3}\varphi^3 + (2k-c)\varphi - \frac{1}{2}y^2}{\varphi - c}. \end{cases} \tag{34}$$

We want to draw the bifurcation phase portraits of (34). But the line $\varphi = c$ bring inconvenience to us. For avoiding the inconvenience temporarily, we make transformation

$$d\tau = \frac{d\xi}{\varphi - c}. \tag{35}$$

Under the transformation (35), system (34) becomes

$$\begin{cases} \frac{d\varphi}{d\tau} = y(\varphi - c), \\ \frac{dy}{d\tau} = \frac{a}{3}\varphi^3 + (2k - c)\varphi - \frac{1}{2}y^2. \end{cases} \tag{36}$$

Since both (34) and (36) have the same first integral

$$y^2(\varphi - c) - \frac{a}{6}\varphi^4 - (2k - c)\varphi^2 = h, \tag{37}$$

the two systems have the same topological phase portraits except the line $\varphi = c$. Through qualitative analysis, we draw the bifurcation phase portraits as Figs. 3–5.

3. The derivations of main results

Substituting the expressions of $u_i(x,t)$ $(i = 1 - 8)$ and $-u_1(x,t)$, $-u_3(x,t)$, $-u_5(x,t)$, $-u_7(x,t)$ and their parametric conditions into Eq. (5), it is not difficult to see that these expressions are solutions of Eq. (5) by using mathematical software Maple. Now we give the derivations of these expressions and show the relations among them. From (8) one see that α is defined in (A_i), (B_i) $(i = 1 - 4)$ and l_j

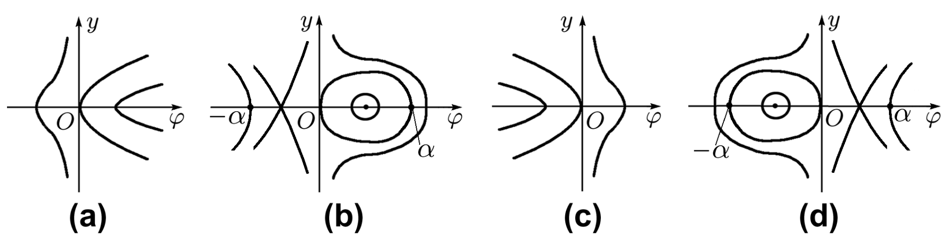

Fig. 3. Bifurcation phase portraits of system (34) and (36) when $c = 0$. (a) $a > 0$ and $k \geqslant 0$; (b) $a < 0$ and $k > 0$; (c) $a < 0$ and $k \leqslant 0$; (d) $a > 0$ and $k < 0$.

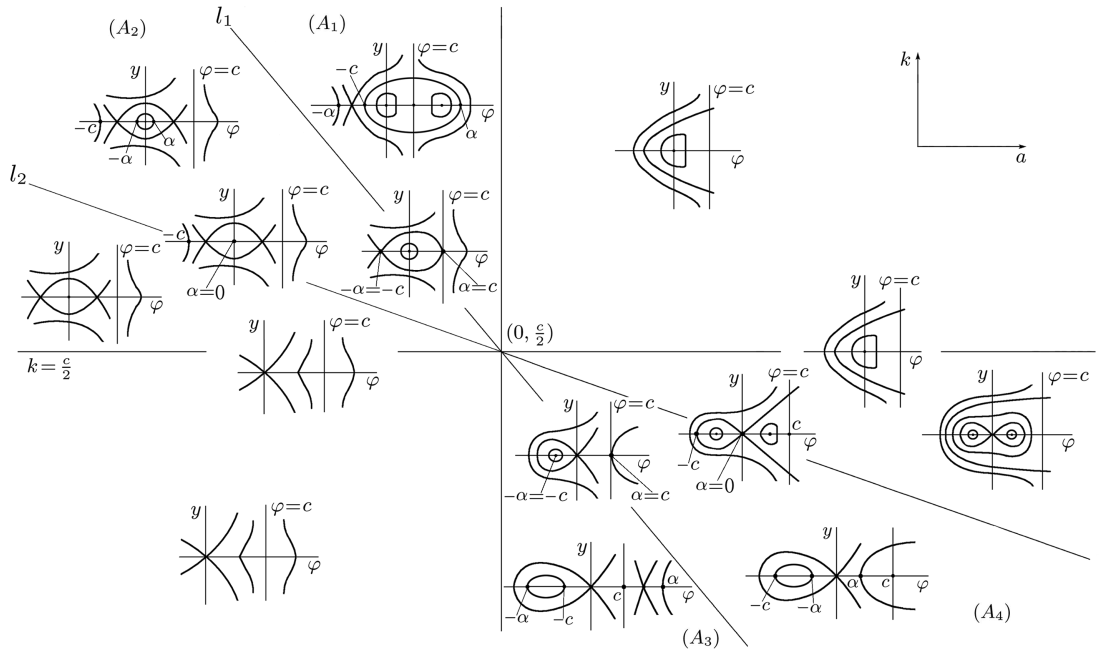

Fig. 4. Bifurcation phase portraits of system (34) and (36) when $c > 0$.

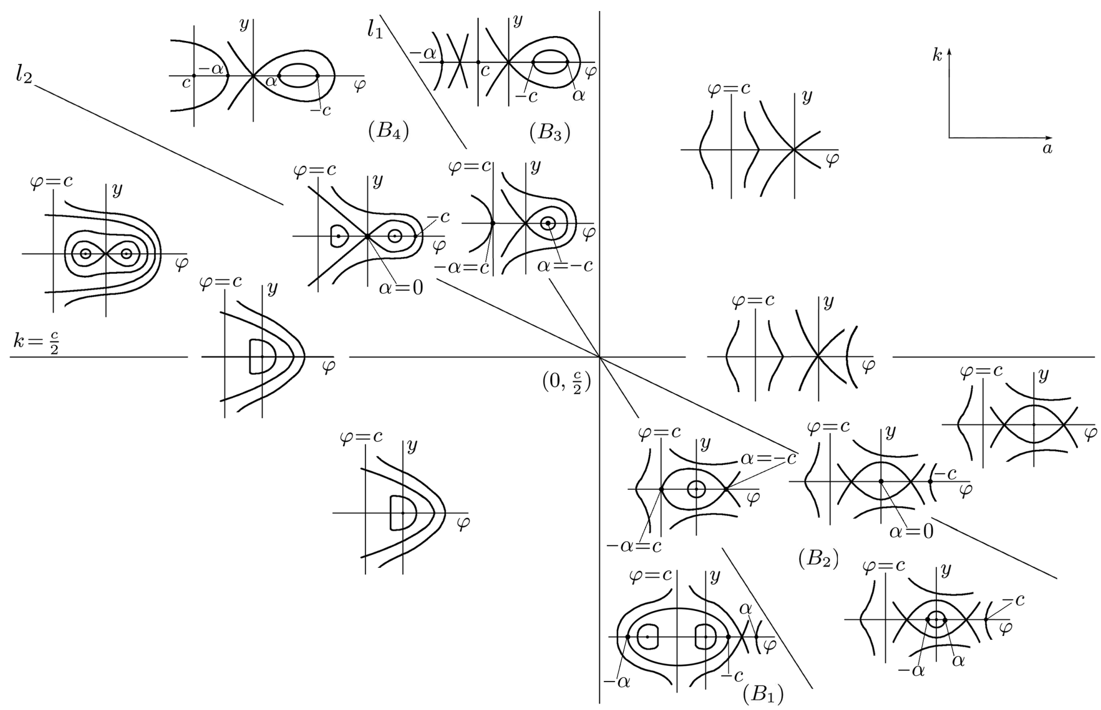

Fig. 5. Bifurcation phase portraits of system (34) and (36) when $c < 0$.

$(j = 1, 2)$. The locations of $-\alpha$, $-c$, c and α are marked in Figs. 3–5.

3.1. The derivations of Proposition 1

For given $c > 0$, from (37) and Fig. 4, we get the expressions of some special orbits of system (34) and their corresponding integral equations as follows.

(1) When $(a, k) \in (A_1)$, the orbit passing point $(-\alpha, 0)$ has expression

$$y = \pm[|a|(\alpha - \varphi)(-c - \varphi)(-\alpha - \varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant -\alpha. \tag{38}$$

Substituting the expression into $\frac{d\varphi}{dy} = d\xi$ and integrating along the orbit, we get its corresponding integral equation

$$\int_{-\infty}^{\varphi} \frac{ds}{\sqrt{(\alpha-s)(-c-s)(-\alpha-s)}} = \sqrt{\frac{|a|}{6}}|\xi| \quad \text{where} \quad \varphi \leqslant -\alpha < -c < \alpha. \tag{39}$$

Similarly we have:

(**2**) When $a<0$ and $(a,k)\in l_1$, the orbit passing point $(-c,0)$ has expression

$$y = \pm(-c-\varphi)[|a|(c-\varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant -c, \tag{40}$$

and its corresponding integral equation

$$\int_{-\infty}^{\varphi} \frac{ds}{(-c-s)\sqrt{c-s}} = \sqrt{\frac{|a|}{6}}|\xi|. \tag{41}$$

(**3**) When $(a,k)\in(A_2)$, the orbit passing point $(-c,0)$ has expression

$$y = \pm[|a|(\alpha-\varphi)(-\alpha-\varphi)(-c-\varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant -c, \tag{42}$$

and its corresponding integral equation

$$\int_{-\infty}^{\varphi} \frac{ds}{\sqrt{(\alpha-s)(-\alpha-s)(-c-s)}} = \sqrt{\frac{|a|}{6}}|\xi| \quad \text{where} \quad \varphi \leqslant -c < -\alpha < \alpha. \tag{43}$$

(**4**) When $a<0$ and $(a,k)\in l_2$, the orbit passing point $(-c,0)$ has expression

$$y = \pm\varphi[|a|(-c-\varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant -c, \tag{44}$$

and its corresponding integral equation

$$\int_{-\infty}^{\varphi} \frac{ds}{s\sqrt{-c-s}} = -\sqrt{\frac{|a|}{6}}|\xi|. \tag{45}$$

(**5**) When $(a,k)\in(A_3)$, the orbit passing point $(\alpha,0)$ has expression

$$y = \pm[a(\varphi-\alpha)(\varphi+c)(\varphi+\alpha)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant \alpha, \tag{46}$$

and its corresponding integral equation

$$\int_{\varphi}^{+\infty} \frac{ds}{\sqrt{(s-\alpha)(s+c)(s+\alpha)}} = \sqrt{\frac{a}{6}}|\xi| \quad \text{where} \quad \varphi \geqslant \alpha > -c > -\alpha. \tag{47}$$

(**6**) When $a>0$ and $(a,k)\in l_1$, the orbit passing point $(c,0)$ has expression

$$y = \pm(\varphi+c)[a(\varphi-c)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant c, \tag{48}$$

and its corresponding integral equation

$$\int_{\varphi}^{+\infty} \frac{ds}{(s+c)\sqrt{s-c}} = \sqrt{\frac{a}{6}}|\xi|. \tag{49}$$

(**7**) When $(a,k)\in(A_4)$, the orbit passing point $(\alpha,0)$ has expression

$$y = \pm[a(\varphi-\alpha)(\varphi+\alpha)(\varphi+c)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant \alpha, \tag{50}$$

and its corresponding integral equation

$$\int_{\varphi}^{+\infty} \frac{ds}{\sqrt{(s-\alpha)(s+\alpha)(s+c)}} = \sqrt{\frac{a}{6}}|\xi| \quad \text{where} \quad \varphi \geqslant \alpha > -\alpha > -c. \tag{51}$$

(**8**) When $a>0$ and $(a,k)\in l_2$, the orbit passing point $(0,0)$ has expression

$$y = \pm\varphi[a(\varphi+c)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant 0, \tag{52}$$

and its corresponding integral equation

$$\int_{\varphi}^{+\infty} \frac{ds}{s\sqrt{s+c}} = \sqrt{\frac{a}{6}}|\xi|. \tag{53}$$

Completing the integral in (39) it follows that

$$\mathrm{sn}^{-1}\left(\sqrt{\frac{2\alpha}{\alpha-\varphi}},\ k_1\right) = \sqrt{\frac{\alpha|a|}{12}}|\xi|, \tag{54}$$

that is

$$\sqrt{\frac{2\alpha}{\alpha-\varphi}} = \mathrm{sn}\left(\sqrt{\frac{\alpha|a|}{12}}, k_1\right), \tag{55}$$

where

$$k_1 = \sqrt{\frac{\alpha+c}{2\alpha}} \quad \text{for} \quad c>0. \tag{56}$$

Solving Eq. (55) yields

$$\varphi = \alpha\left[1 - 2\mathrm{sn}^{-2}\left(\sqrt{\frac{\alpha|a|}{12}}\xi, k_1\right)\right]. \tag{57}$$

From (31) and $u=\varphi(\xi)$, we obtain the periodic blow-up solution $u_1(x,t)$ as (13).

Similarly, completing the integrals in (41), (43), (45), (47), (49), (51), (53) and solving the equations for φ, respectively, we get $u_i(x,t)$ $(i=2,\cdots,8)$ as (15), (16), (18), (19), (21), (22) and (24). These complete the derivations of Proposition 1.

3.2. The derivations of Proposition 2

For given $c<0$, via (37) and Fig. 5, we obtain the expressions of some special orbits of system (34) as follows.

(**1**) When $(a,k)\in(B_1)$, the orbit passing point $(\alpha,0)$ has expression

$$y = \pm[a(\varphi-\alpha)(\varphi+c)(\varphi+\alpha)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant \alpha. \tag{58}$$

(**2**) When $a>0$ and $(a,k)\in l_1$, the orbit passing point $(-c,0)$ has expression

$$y = \pm(\varphi+c)[a(\varphi-c)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant -c. \tag{59}$$

(**3**) When $(a,k)\in(B_2)$, the orbit passing point $(-c,0)$ has expression

$$y = \pm[a(\varphi + c)(\varphi - \alpha)(\varphi + \alpha)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant -c. \qquad (60)$$

(4) When $a > 0$ and $(a, k) \in l_2$, the orbit passing point $(-c, 0)$ has expression

$$y = \pm\varphi[a(\varphi + c)/6]^{1/2} \quad \text{for} \quad \varphi \geqslant -c. \qquad (61)$$

(5) When $(a, k) \in (B_3)$, the orbit passing point $(-\alpha, 0)$ has expression

$$y = \pm[|a|(\alpha - \varphi)(-c - \varphi)(-\alpha - \varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant -\alpha. \qquad (62)$$

(6) When $a < 0$ and $(a, k) \in l_1$, the orbit passing point $(c, 0)$ has expression

$$y = \pm(-c - \varphi)[|a|(c - \varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant c. \qquad (63)$$

(7) When $(a, k) \in (B_4)$, the orbit passing point $(-\alpha, 0)$ has expression

$$y = \pm[|a|(-c - \varphi)(\alpha - \varphi)(-\alpha - \varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant -\alpha. \qquad (64)$$

(8) When $a < 0$ and $(a, k) \in l_2$, the orbit passing point $(0, 0)$ has expression

$$y = \pm\varphi[|a|(-c - \varphi)/6]^{1/2} \quad \text{for} \quad \varphi \leqslant 0. \qquad (65)$$

Similar to the derivations of Proposition 1, using the expressions above to establish integral equations, then solving the integral equations for φ, we get the conclusions of Proposition 2.

3.3. *The proof of Property 1*

For given $c > 0$, we have:

(1) When $(a, k) \in (A_1)$ and tends to l_1, from (6), (8)–(10), (14) it follows that

$$\alpha \to c, \ \beta_1 \to \beta_2, \ k_1 \to 1 \ \text{and} \ \text{sn}(z, 1) = \tanh z. \qquad (66)$$

Via (13) and (66), one can see that $u_1(x, t)$ becomes $u_2(x, t)$ when $(a, k) \in (A_1)$ and tends to l_1. This completes the derivation of relation (1°).

(2) When $(a, k) \in (A_2)$ and tends to l_1, from (6), (8), (10), (11), (17) it follows that

$$\alpha \to c, \ \beta_3 \to \beta_2, \ k_2 \to 1 \ \text{and} \ \text{sn}(z, 1) = \tanh z. \qquad (67)$$

Through (16) and (67), one can see that $u_3(x, t)$ becomes $u_2(x, t)$ when $(a, k) \in (A_2)$ and tends to l_1.
On the other hand, when $(a, k) \in (A_2)$ and tends to l_2, from (7), (11), (12), (17) it follows that

$$\alpha \to 0, \ \beta_3 \to \beta_4, \ k_2 \to 0 \ \text{and} \ \text{sn}(z, 0) = \sin z. \qquad (68)$$

From (16) and (68), one can see that $u_3(x, t)$ becomes $u_4(x, t)$ when $(a, k) \in (A_2)$ and tends to l_2. This completes the derivation of relation (2°).

(3) When $(a, k) \in (A_3)$ and tends to l_1, from (6), (8)–(10), (20) it follows that

$$\alpha \to c, \ \beta_1 \to \beta_2, \ k_3 \to 0 \ \text{and} \ \text{sn}(z, 0) = \sin z. \qquad (69)$$

Via (19) and (69), one can see that $u_5(x, t)$ becomes $u_6(x, t)$ when $(a, k) \in (A_3)$ and tends to l_1. This implies the correctness of relation (3°).

(4) When $(a, k) \in (A_4)$ and tends to l_1, from (6), (8), (10), (11), (23) it follows that

$$\alpha \to c, \ \beta_3 \to \beta_2, \ k_4 \to 0 \ \text{and} \ \text{sn}(z, 0) = \sin z. \qquad (70)$$

Via (22) and (70), one can see that $u_7(x, t)$ becomes $u_6(x, t)$ when $(a, k) \in (A_4)$ and tends to l_1.
On the other hand, when $(a, k) \in (A_4)$ and tends to l_2, from (7), (8), (11), (12), (23) it follows that

$$\alpha \to 0, \ \beta_3 \to \beta_4, \ k_4 \to 1 \ \text{and} \ \text{sn}(z, 1) = \tanh z. \qquad (71)$$

Via (22) and (71), one can see that $u_7(x, t)$ becomes $u_8(x, t)$ when $(a, k) \in (A_4)$ and tends to l_2. These show the correctness of relation (4°). About the relations (1*)–(4*) given in Property 2, the proof is similar to that above. Here, we would not repeat it.

Remark 3. When $c = 0$, the stationary solutions can be obtained via Fig. 3(a)–(d).

4. Conclusion

In this paper, we considered Eq. (5). We obtained some new periodic wave solutions and their limit forms which were given in Propositions 1, 2. One can see that the expressions of these solutions are very simple and the periodic wave solutions tend to infinity on $\xi - u$ plane periodically. To our knowledge, such solutions have not been found in any other equations.

From previous results (see Ref. [23–25]) and our results above, one can see that in Eq. (1) the effect of changing the convection term uu_x to u^2u_x causes not only the coexistence of bell-shaped solitary wave solution and peakon solution, but also the appearance of periodic blow-up solutions. We think that Eq. (5) should have more complex phenomena waiting for discovery.

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